Local Complexity Zoo

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This is a short guide to the complexity classes and concepts in-scope for class and problem sets. Languages are sets of strings and complexity classes are sets of languages.

1 Inclusions & Separations: Infinitely-Often Classes (i.o.$\mathcal{C}$)

Often, we will want to decompose a language $\mathcal{L} \subseteq \{0,1\}^*$ into n-bit *slices* $\mathcal{L}_n = \mathcal{L} \cap \{0,1\}^n$. Given a complexity class $\mathcal{C}$, i.o. $\mathcal{C}$ is the class of languages $\mathcal{L}$ for which there is a language $\mathcal{L}' \in \mathcal{C}$ such that $\mathcal{L}_n = \mathcal{L}'_n$ for infinitely many input lengths n. This is a way to make inclusions *weaker* — if $\mathsf{SAT} \in$ i.o P then this could be quite useless. It could be that the input lengths n for which the SAT algorithm agrees with SAT are too far apart to matter. It is also a way to make separations *stronger* — if complexity class $\mathcal{D} \not\subseteq$ i.o. $\mathcal{C}$ then at **every** input length past a certain point, there exists an error witnessing that some langauge $H \in \mathcal{D}$ is not decicable in $\mathcal{C}$. For an extended discussion of these issues see the introduction of [FS17].

2 Problems

Definition 1 (Minimum Circuit Size Problem [KC00])**.** Given as input the truth table of an n-bit Boolean function f and a number $s \in \{1, \ldots, 2^n\}$, accept if f can be computed by a Boolean circuit with at most s gates.

3 Complexity Classes

Definition 2 (Zero-Error Probabilistic Time (Ex 7.6 of [AB09]))**.** A language $\mathcal{L}$ is in $\mathsf{ZPTIME}[T(n)]$ iff there exists a $T(n)$-time probabilistic TM M with outputs in $\{0, 1, ?\}$ such that, for every input, if it answers with 0 or 1 the answer is correct, and the probability of answering "I don't know" $(= ?)$ is bounded:

$$\forall x \in \{0,1\}^* \quad \Pr[M(x) \in \{\mathcal{L}(x), ?\}] = 1 \wedge \Pr[M(x) = ?] \leq 1/2$$

References

[AB09] Sanjeev Arora and Boaz Barak. *Computational Complexity - A Modern Approach.* Cambridge University Press, 2009. ISBN: 978-0-521-42426-4. URL: `http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521424264`.

[FS17] Lance Fortnow and Rahul Santhanam. "Robust simulations and significant separations". In: *Inf. Comput.* 256 (2017), pp. 149–159. DOI: `10.1016/j.ic.2017.07.002`. URL: `https://doi.org/10.1016/j.ic.2017.07.002`.

[KC00] Valentine Kabanets and Jin-yi Cai. "Circuit minimization problem". In: *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*. Ed. by F. Frances Yao and Eugene M. Luks. ACM, 2000, pp. 73–79. DOI: `10.1145/335305.335314`. URL: `https://doi.org/10.1145/335305.335314`.